Chapter 3

FORWARD KINEMATICS: THE DENAVIT-HARTENBERG CONVENTION

In this chapter we develop the **forward** or **configuration kinematic equations** for rigid robots. The forward kinematics problem is concerned with the relationship between the individual joints of the robot manipulator and the position and orientation of the tool or end-effector. Stated more formally, the forward kinematics problem is to determine the position and orientation of the end-effector, given the values for the joint variables of the robot. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension in the case of prismatic or sliding joints. The forward kinematics problem is to be contrasted with the **inverse** kinematics problem, which will be studied in the next chapter, and which is concerned with determining values for the joint variables that achieve a desired position and orientation for the end-effector of the robot.

3.1 Kinematic Chains

As described in Chapter 1, a robot manipulator is composed of a set of links connected together by various joints. The joints can either be very simple, such as a revolute joint or a prismatic joint, or else they can be more complex, such as a ball and socket joint. (Recall that a revolute joint is like a hinge and allows a relative rotation about a single axis, and a prismatic joint permits a linear motion along a single axis, namely an extension or retraction.) The difference between the two situations is that, in the first instance, the joint has only a single degree-of-freedom of motion: the angle of rotation in the case of a revolute joint, and the amount of linear displacement in the case of a prismatic joint. In contrast, a ball and socket joint has two degrees-of-freedom. In this book it is assumed throughout that all joints have only a single degree-of-freedom. Note that the assumption does not involve any real loss of generality, since joints such as a ball and socket joint (two degrees-of-freedom) or a spherical wrist (three degrees-of-freedom) can always be thought of as a succession of single degree-of-freedom joints with links of length zero in between.

With the assumption that each joint has a single degree-of-freedom, the action of each joint can be described by a single real number: the angle of rotation in the case of a revolute joint or the displacement in the case of a prismatic joint. The objective of forward kinematic analysis is to determine the *cumulative* effect of the entire set of joint variables. In this chapter we will develop a set of conventions that provide a systematic procedure for performing this analysis. It is, of course, possible to carry out forward kinematics analysis even without respecting these conventions, as we did for the two-link planar manipulator example in Chapter 1. However, the kinematic analysis of an n-link manipulator can be extremely complex and the conventions introduced below simplify the analysis considerably. Moreover, they give rise to a universal language with which robot engineers can communicate.

A robot manipulator with n joints will have $n+1$ links, since each joint connects two links. We number the joints from 1 to n, and we number the links from 0 to n, starting from the base. By this convention, joint i connects link $i-1$ to link i. We will consider the location of joint i to be fixed with respect to link $i-1$. *When joint i is actuated, link i moves.* Therefore, link 0 (the first link) is fixed, and does not move when the joints are actuated. Of course the robot manipulator could itself be mobile (e.g., it could be mounted on a mobile platform or on an autonomous vehicle), but we will not consider this case in the present chapter, since it can be handled easily by slightly extending the techniques presented here.

With the i^{th} joint, we associate a *joint variable*, denoted by q_i. In the case of a revolute joint, q_i is the angle of rotation, and in the case of a

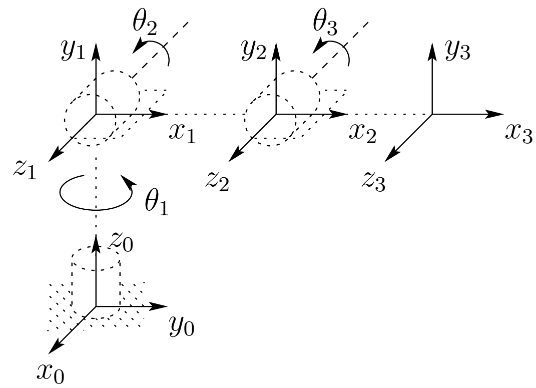

Figure 3.1: Coordinate frames attached to elbow manipulator.

prismatic joint, q_i is the joint displacement:

$$q_i = \begin{cases} \theta_i & : \text{ joint i revolute} \\ d_i & : \text{ joint i prismatic} \end{cases} . \tag{3.1}$$

To perform the kinematic analysis, we rigidly attach a coordinate frame to each link. In particular, we attach $o_ix_iy_iz_i$ to link i. This means that, whatever motion the robot executes, the coordinates of each point on link i are constant when expressed in the i^{th} coordinate frame. Furthermore, when joint i is actuated, link i and its attached frame, $o_ix_iy_iz_i$, experience a resulting motion. The frame $o_0x_0y_0z_0$, which is attached to the robot base, is referred to as the inertial frame. Figure 3.1 illustrates the idea of attaching frames rigidly to links in the case of an elbow manipulator.

Now suppose A_i is the homogeneous transformation matrix that expresses the position and orientation of $o_ix_iy_iz_i$ with respect to $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. The matrix A_i is not constant, but varies as the configuration of the robot is changed. However, the assumption that all joints are either revolute or prismatic means that A_i is a function of only a single joint variable, namely q_i. In other words,

$$A_i \quad = \quad A_i(q_i). \tag{3.2}$$

Now the homogeneous transformation matrix that expresses the position and orientation of $o_jx_jy_jz_j$ with respect to $o_ix_iy_iz_i$ is called, by convention, a **transformation matrix**, and is denoted by T^i_j. From Chapter 2 we see that

$$T^i_j \quad = \quad A_{i+1}A_{i+2}\dots A_{j-1}A_j \text{ if } i < j$$

$$\begin{aligned} T_j^i &= I \text{ if } i = j \\ T_j^i &= (T_i^j)^{-1} \text{ if } j > i. \end{aligned} \tag{3.3}$$

By the manner in which we have rigidly attached the various frames to the corresponding links, it follows that the position of any point on the end-effector, when expressed in frame n, is a constant independent of the configuration of the robot. Denote the position and orientation of the end-effector with respect to the inertial or base frame by a three-vector O_n^0 (which gives the coordinates of the origin of the end-effector frame with respect to the base frame) and the 3×3 rotation matrix R_n^0, and define the homogeneous transformation matrix

$$H = \begin{bmatrix} R_n^0 & O_n^0 \\ 0 & 1 \end{bmatrix}. \tag{3.4}$$

Then the position and orientation of the end-effector in the inertial frame are given by

$$H = T_n^0 = A_1(q_1) \cdots A_n(q_n). \tag{3.5}$$

Each homogeneous transformation A_i is of the form

$$A_i = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix}. \tag{3.6}$$

Hence

$$T_j^i = A_{i+1} \cdots A_j = \begin{bmatrix} R_j^i & O_j^i \\ 0 & 1 \end{bmatrix}. \tag{3.7}$$

The matrix R_j^i expresses the orientation of $o_j x_j y_j z_j$ relative to $o_i x_i y_i z_i$ and is given by the rotational parts of the A-matrices as

$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}. \tag{3.8}$$

The coordinate vectors O_j^i are given recursively by the formula

$$O_j^i = O_{j-1}^i + R_{j-1}^i O_j^{j-1}, \tag{3.9}$$

These expressions will be useful in Chapter 5 when we study Jacobian matrices.

In principle, that is all there is to forward kinematics! Determine the functions $A_i(q_i)$, and multiply them together as needed. However, it is possible to achieve a considerable amount of streamlining and simplification by introducing further conventions, such as the Denavit-Hartenberg representation of a joint, and this is the objective of the remainder of the chapter.

3.2 Denavit Hartenberg Representation

While it is possible to carry out all of the analysis in this chapter using an arbitrary frame attached to each link, it is helpful to be systematic in the choice of these frames. A commonly used convention for selecting frames of reference in robotic applications is the Denavit-Hartenberg, or D-H convention. In this convention, each homogeneous transformation A_i is represented as a product of four basic transformations

$$\begin{aligned} A_i &= R_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}R_{x,\alpha_i} \qquad (3.10) \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where the four quantities θ_i, a_i, d_i, α_i are parameters associated with link i and joint i. The four parameters a_i, α_i, d_i, and θ_i in (3.10) are generally given the names **link length**, **link twist**, **link offset**, and **joint angle**, respectively. These names derive from specific aspects of the geometric relationship between two coordinate frames, as will become apparent below. Since the matrix A_i is a function of a single variable, it turns out that three of the above four quantities are constant for a given link, while the fourth parameter, θ_i for a revolute joint and d_i for a prismatic joint, is the joint variable.

From Chapter 2 one can see that an arbitrary homogeneous transformation matrix can be characterized by six numbers, such as, for example, three numbers to specify the fourth column of the matrix and three Euler angles to specify the upper left 3×3 rotation matrix. In the D-H representation, in contrast, there are only *four* parameters. How is this possible? The answer is that, while frame i is required to be rigidly attached to link i, we have considerable freedom in choosing the origin and the coordinate axes of the frame. For example, it is not necessary that the origin, O_i, of frame i be placed at the physical end of link i. In fact, it is not even necessary that frame i be placed within the physical link; frame i could lie in free space — so long as frame i is *rigidly attached* to link i. By a clever choice of the origin and the coordinate axes, it is possible to cut down the number of parameters

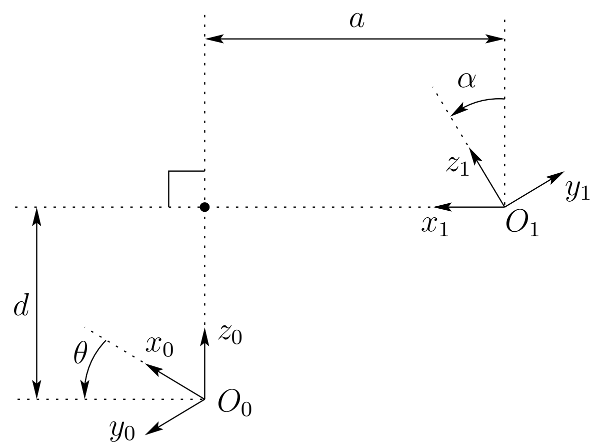

Figure 3.2: Coordinate frames satisfying assumptions DH1 and DH2.

needed from six to four (or even fewer in some cases). In Section 3.2.1 we will show why, and under what conditions, this can be done, and in Section 3.2.2 we will show exactly how to make the coordinate frame assignments.

3.2.1 Existence and uniqueness issues

Clearly it is not possible to represent any arbitrary homogeneous transformation using only four parameters. Therefore, we begin by determining just which homogeneous transformations can be expressed in the form (3.10). Suppose we are given two frames, denoted by frames 0 and 1, respectively. Then there exists a unique homogeneous transformation matrix A that takes the coordinates from frame 1 into those of frame 0. Now suppose the two frames have two additional features, namely:

(DH1) The axis x_1 is perpendicular to the axis z_0

(DH2) The axis x_1 intersects the axis z_0

as shown in Figure 3.2. Under these conditions, we claim that there exist unique numbers a, d, θ, α such that

$$A = R_{z,\theta}\text{Trans}_{z,d}\text{Trans}_{x,a}R_{x,\alpha}. \quad (3.11)$$

Of course, since θ and α are angles, we really mean that they are unique to within a multiple of 2π. To show that the matrix A can be written in this form, write A as

$$A = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \tag{3.12}$$

and let $\boldsymbol{r}_i$ denote the i^{th} column of the rotation matrix R_1^0. We will now examine the implications of the two DH constraints.

If (DH1) is satisfied, then x_1 is perpendicular to z_0 and we have $x_1 \cdot z_0 = 0$. Expressing this constraint with respect to $o_0 x_0 y_0 z_0$, using the fact that $\boldsymbol{r}_1$ is the representation of the unit vector x_1 with respect to frame 0, we obtain

$$0 = x_1^0 \cdot z_0^0 \tag{3.13}$$
$$= [r_{11}, r_{21}, r_{31}]^T \cdot [0, 0, 1]^T \tag{3.14}$$
$$= r_{31}. \tag{3.15}$$

Since $r_{31} = 0$, we now need only show that there exist *unique* angles θ and α such that

$$R_1^0 = R_{x,\theta} R_{x,\alpha} = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}. \tag{3.16}$$

The only information we have is that $r_{31} = 0$, but this is enough. First, since each row and column of R_1^0 must have unit length, $r_{31} = 0$ implies that

$$r_{11}^2 + r_{21}^2 = 1,$$
$$r_{32}^2 + r_{33}^2 = 1 \tag{3.17}$$

Hence there exist unique θ, α such that

$$(r_{11}, r_{21}) = (c_\theta, s_\theta), \qquad (r_{33}, r_{32}) = (c_\alpha, s_\alpha). \tag{3.18}$$

Once θ and α are found, it is routine to show that the remaining elements of R_1^0 must have the form shown in (3.16), using the fact that R_1^0 is a rotation matrix.

Next, assumption (DH2) means that the displacement between O_0 and O_1 can be expressed as a linear combination of the vectors z_0 and x_1. This can be written as $O_1 = O_0 + dz_0 + ax_1$. Again, we can express this relationship in the coordinates of $o_0 x_0 y_0 z_0$, and we obtain

$$O_1^0 = O_0^0 + dz_0^0 + ax_1^0 \tag{3.19}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix} \tag{3.20}$$

$$= \begin{bmatrix} ac_\theta \\ as_\theta \\ d \end{bmatrix}. \tag{3.21}$$

Combining the above results, we obtain (3.10) as claimed. Thus, we see that four parameters are sufficient to specify any homogeneous transformation that satisfies the constraints (DH1) and (DH2).

Now that we have established that each homogeneous transformation matrix satisfying conditions (DH1) and (DH2) above can be represented in the form (3.10), we can in fact give a physical interpretation to each of the four quantities in (3.10). The parameter a is the distance between the axes z_0 and z_1, and is measured along the axis x_1. The angle α is the angle between the axes z_0 and z_1, measured in a plane normal to x_1. The positive sense for α is determined from z_0 to z_1 by the right-hand rule as shown in Figure 3.3. The parameter d is the distance between the origin

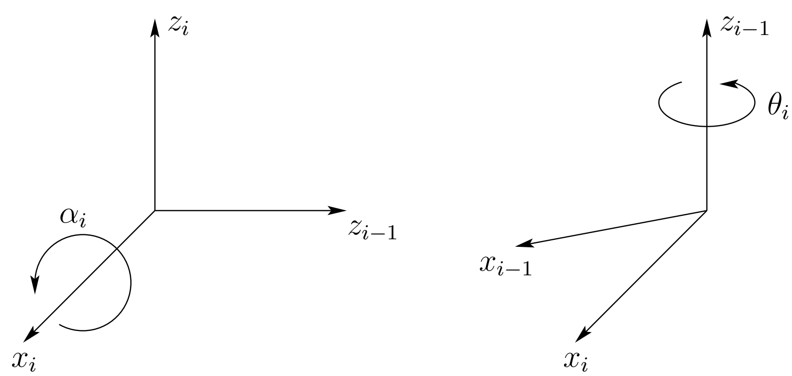

Figure 3.3: Positive sense for α_i and θ_i.

O_0 and the intersection of the x_1 axis with z_0 measured along the z_0 axis. Finally, θ is the angle between x_0 and x_1 measured in a plane normal to z_0. These physical interpretations will prove useful in developing a procedure for assigning coordinate frames that satisfy the constraints (DH1) and (DH2), and we now turn our attention to developing such a procedure.

3.2.2 Assigning the coordinate frames

For a given robot manipulator, one can always choose the frames $0, \ldots, n$ in such a way that the above two conditions are satisfied. In certain circumstances, this will require placing the origin O_i of frame i in a location that may not be intuitively satisfying, but typically this will not be the case. In reading the material below, it is important to keep in mind that the choices of the various coordinate frames are not unique, even when constrained by the requirements above. Thus, it is possible that different engineers will derive differing, but equally correct, coordinate frame assignments for the links of the robot. It is very important to note, however, that the end result (i.e., the matrix T_n^0) will be the same, regardless of the assignment of intermediate link frames (assuming that the coordinate frames for link n coincide). We will begin by deriving the general procedure. We will then discuss various common special cases where it is possible to further simplify the homogeneous transformation matrix.

To start, note that the choice of z_i is arbitrary. In particular, from (3.16), we see that by choosing α_i and θ_i appropriately, we can obtain any arbitrary direction for z_i. Thus, for our first step, we assign the axes $z_0, \ldots, z_{n-1}$ in an intuitively pleasing fashion. Specifically, we assign z_i to be the axis of actuation for joint $i+1$. Thus, z_0 is the axis of actuation for joint 1, z_1 is the axis of actuation for joint 2, etc. There are two cases to consider: (i) if joint $i+1$ is revolute, z_i is the axis of revolution of joint $i+1$; (ii) if joint $i+1$ is prismatic, z_i is the axis of translation of joint $i+1$. At first it may seem a bit confusing to associate z_i with joint $i+1$, but recall that this satisfies the convention that we established in Section 3.1, namely that joint i is fixed with respect to frame i, and that when joint i is actuated, link i and its attached frame, $o_i x_i y_i z_i$, experience a resulting motion.

Once we have established the z-axes for the links, we establish the base frame. The choice of a base frame is nearly arbitrary. We may choose the origin O_0 of the base frame to be any point on z_0. We then choose x_0, y_0 in any convenient manner so long as the resulting frame is right-handed. This sets up frame 0.

Once frame 0 has been established, we begin an iterative process in which we define frame i using frame $i-1$, beginning with frame 1. Figure 3.4 will be useful for understanding the process that we now describe.

In order to set up frame i it is necessary to consider three cases: (i) the axes z_{i-1}, z_i are not coplanar, (ii) the axes z_{i-1}, z_i intersect (iii) the axes z_{i-1}, z_i are parallel. Note that in both cases (ii) and (iii) the axes z_{i-1} and z_i are coplanar. This situation is in fact quite common, as we will see in

Figure 3.4: Denavit-Hartenberg frame assignment.

Section 3.3. We now consider each of these three cases.

(i) z_{i-1} and z_i are not coplanar: If z_{i-l} and z_i are not coplanar, then there exists a *unique* line segment perpendicular to both z_{i-1} and z_i such that it connects both lines and it has minimum length. The line containing this common normal to z_{i-1} and z_i defines x_i, and the point where this line intersects z_i is the origin O_i. By construction, both conditions (DH1) and (DH2) are satisfied and the vector from O_{i-1} to O_i is a linear combination of z_{i-1} and x_i. The specification of frame i is completed by choosing the axis y_i to form a right-hand frame. Since assumptions (DH1) and (DH2) are satisfied the homogeneous transformation matrix A_i is of the form (3.10).

(ii) z_{i-1} is parallel to z_i: If the axes z_{i-1} and z_i are parallel, then there are infinitely many common normals between them and condition (DH1) does not specify x_i completely. In this case we are free to choose the origin O_i anywhere along z_i. One often chooses O_i to simplify the resulting equations. The axis x_i is then chosen either to be directed from O_i toward z_{i-1}, along the common normal, or as the opposite of this vector. A common method for choosing O_i is to choose the normal that passes through O_{i-1} as the x_i axis; O_i is then the point at which this normal intersects z_i. In this case, d_i would be equal to zero. Once x_i is fixed, y_i is determined, as usual by the right hand rule. Since the axes z_{i-1} and z_i are parallel, α_i will be zero in this case.

(iii) z_{i-1} intersects z_i: In this case x_i is chosen normal to the plane formed by z_i and z_{i-1}. The positive direction of x_i is arbitrary. The most natural choice for the origin O_i in this case is at the point of intersection of z_i and z_{i-1}. However, any convenient point along the axis z_i suffices. Note that in this case the parameter a_i equals 0.

This constructive procedure works for frames $0, \ldots, n-l$ in an n-link robot. To complete the construction, it is necessary to specify frame n. The final coordinate system $o_n x_n y_n z_n$ is commonly referred to as the **end-effector** or **tool frame** (see Figure 3.5). The origin O_n is most often

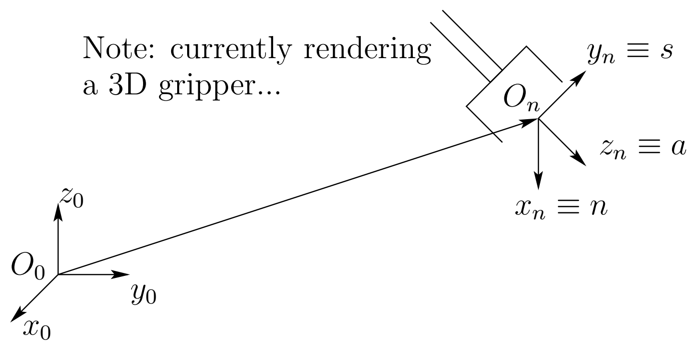

Figure 3.5: Tool frame assignment.

placed symmetrically between the fingers of the gripper. The unit vectors along the x_n, y_n, and z_n axes are labeled as $\boldsymbol{n}$, $\boldsymbol{s}$, and $\boldsymbol{a}$, respectively. The terminology arises from fact that the direction $\boldsymbol{a}$ is the **approach** direction, in the sense that the gripper typically approaches an object along the $\boldsymbol{a}$ direction. Similarly the $\boldsymbol{s}$ direction is the **sliding** direction, the direction along which the fingers of the gripper slide to open and close, and $\boldsymbol{n}$ is the direction **normal** to the plane formed by $\boldsymbol{a}$ and $\boldsymbol{s}$.

In contemporary robots the final joint motion is a rotation of the end-effector by θ_n and the final two joint axes, z_{n-1} and z_n, coincide. In this case, the transformation between the final two coordinate frames is a translation along z_{n-1} by a distance d_n followed (or preceded) by a rotation of θ_n radians about z_{n-1}. This is an important observation that will simplify the computation of the inverse kinematics in the next chapter.

Finally, note the following important fact. In all cases, whether the joint in question is revolute or prismatic, the quantities a_i and α_i are always constant for all i and are characteristic of the manipulator. If joint i is prismatic, then θ_i is also a constant, while d_i is the i^{th} joint variable. Similarly, if joint i is revolute, then d_i is constant and θ_i is the i^{th} joint variable.

3.2.3 Summary

We may summarize the above procedure based on the D-H convention in the following algorithm for deriving the forward kinematics for any manipulator.

Step l: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin O_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate O_i at this intersection. If z_i and z_{i-1} are parallel, locate O_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through O_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin O_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from O_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from O_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i (see Figure 3.3).

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 3.3). θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

3.3 Examples

In the D-H convention the only variable angle is θ, so we simplify notation by writing c_i for $\cos\theta_i$, etc. We also denote $\theta_1 + \theta_2$ by θ_{12}, and $\cos(\theta_1 + \theta_2)$ by c_{12}, and so on. In the following examples it is important to remember that the D-H convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

Example 3.1 Planar Elbow Manipulator

Consider the two-link planar arm of Figure 3.6. The joint axes z_0 *and*

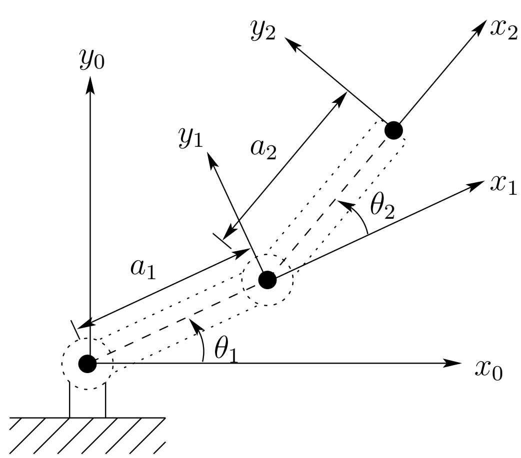

Figure 3.6: Two-link planar manipulator. The z-axes all point out of the page, and are not shown in the figure.

z_1 *are normal to the page. We establish the base frame* $o_0x_0y_0z_0$ *as shown. The origin is chosen at the point of intersection of the* z_0 *axis with the page and the direction of the* x_0 *axis is completely arbitrary. Once the base frame is established, the* $o_1x_1y_1z_1$ *frame is fixed as shown by the D-H convention, where the origin* O_1 *has been located at the intersection of* z_1 *and the page. The final frame* $o_2x_2y_2z_2$ *is fixed by choosing the origin* O_2 *at the end of link 2 as shown. The link parameters are shown in Table 3.1. The A-matrices are*

Table 3.1: Link parameters for 2-link planar manipulator.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

determined from (3.10) as

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.22}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.23}$$

The T-matrices are thus given by

$$T_1^0 = A_1. \tag{3.24}$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.25}$$

Notice that the first two entries of the last column of T_2^0 are the x and y components of the origin O_2 in the base frame; that is,

$$\begin{aligned} x &= a_1c_1 + a_2c_{12} \\ y &= a_1s_1 + a_2s_{12} \end{aligned} \tag{3.26}$$

are the coordinates of the end-effector in the base frame. The rotational part of T_2^0 gives the orientation of the frame $o_2x_2y_2z_2$ relative to the base frame.
◇

Example 3.2 Three-Link Cylindrical Robot

Consider now the three-link cylindrical robot represented symbolically by Figure 3.7. We establish O_0 as shown at joint 1. Note that the placement of

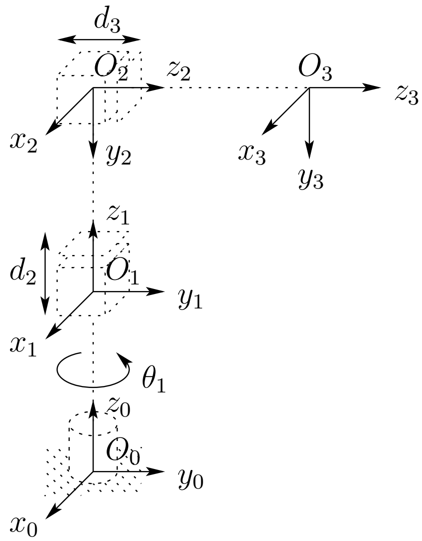

Figure 3.7: Three-link cylindrical manipulator.

Table 3.2: Link parameters for 3-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

the origin O_0 along z_0 as well as the direction of the x_0 axis are arbitrary. Our choice of O_0 is the most natural, but O_0 could just as well be placed at joint 2. The axis x_0 is chosen normal to the page. Next, since z_0 and z_1 coincide, the origin O_1 is chosen at joint 1 as shown. The x_1 axis is normal to the page when $\theta_1 = 0$ but, of course its direction will change since θ_1 is variable. Since z_2 and z_1 intersect, the origin O_2 is placed at this intersection. The direction of x_2 is chosen parallel to x_1 so that θ_2 is zero. Finally, the third frame is chosen at the end of link 3 as shown.

The link parameters are now shown in Table 3.2. The corresponding A *and T matrices are*

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.27}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1A_2A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1d_3 \\ s_1 & 0 & c_1 & c_1d_3 \\ 0 & -1 & 0 & d_1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.28}$$

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Example 3.3 Spherical Wrist

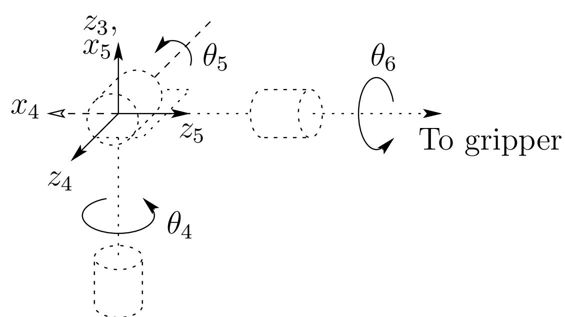

Figure 3.8: The spherical wrist frame assignment.

The spherical wrist configuration is shown in Figure 3.8, in which the joint axes z_3, z_4, z_5 intersect at O. The Denavit-Hartenberg parameters are shown in Table 3.3. The Stanford manipulator is an example of a manipulator that possesses a wrist of this type. In fact, the following analysis applies to virtually all spherical wrists.

Table 3.3: DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

We show now that the final three joint variables, θ_4, θ_5, θ_6 are the Euler angles ϕ, θ, ψ, respectively, with respect to the coordinate frame $o_3x_3y_3z_3$. To see this we need only compute the matrices A_4, A_5, and A_6 using Table 3.3 and the expression (3.10). This gives

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.29}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.30}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.31}$$

Multiplying these together yields

$$T_6^3 = A_4A_5A_6 = \begin{bmatrix} R_6^3 & O_6^3 \\ 0 & 1 \end{bmatrix} \tag{3.32}$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Comparing the rotational part R_6^3 of T_6^3 with the Euler angle transformation (2.51) shows that $\theta_4, \theta_5, \theta_6$ can indeed be identified as the Euler angles ϕ, θ and ψ with respect to the coordinate frame $o_3x_3y_3z_3$.
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Example 3.4 Cylindrical Manipulator with Spherical Wrist

Suppose that we now attach a spherical wrist to the cylindrical manipulator of Example 3.3.2 as shown in Figure 3.9. Note that the axis of rotation of

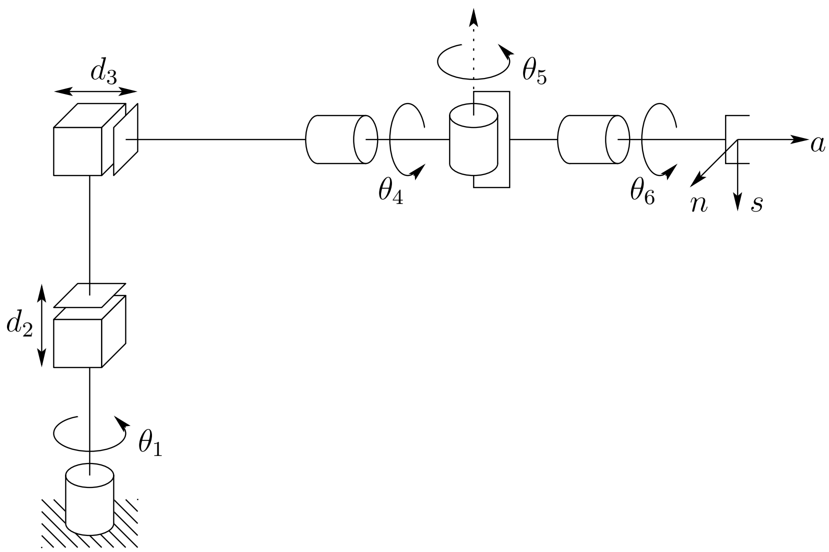

Figure 3.9: Cylindrical robot with spherical wrist.

joint 4 is parallel to z_2 *and thus coincides with the axis* z_3 *of Example 3.3.2. The implication of this is that we can immediately combine the two previous expression (3.28) and (3.32) to derive the forward kinematics as*

$$T_6^0 = T_3^0 T_6^3 \tag{3.33}$$

with T_3^0 *given by (3.28) and* T_6^3 *given by (3.32). Therefore the forward kinematics of this manipulator is described by*

$$\begin{aligned} T_6^0 &= \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_1 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2.
\end{aligned}
$$

Notice how most of the complexity of the forward kinematics for this manipulator results from the orientation of the end-effector while the expression for the arm position from (3.28) is fairly simple. The spherical wrist assumption not only simplifies the derivation of the forward kinematics here, but will also greatly simplify the inverse kinematics problem in the next chapter.

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Example 3.5 Stanford Manipulator

Consider now the Stanford Manipulator shown in Figure 3.10. This manipulator is an example of a spherical (RRP) manipulator with a spherical wrist. This manipulator has an offset in the shoulder joint that slightly complicates both the forward and inverse kinematics problems.

We first establish the joint coordinate frames using the D-H convention as shown. The link parameters are shown in the Table 3.4.

It is straightforward to compute the matrices A_i *as*

$$
A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.35}
$$

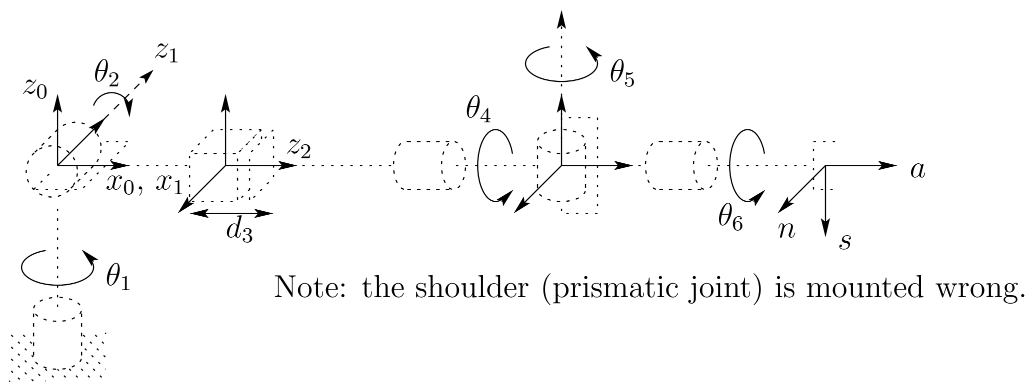

Figure 3.10: DH coordinate frame assignment for the Stanford manipulator.

Table 3.4: DH parameters for Stanford Manipulator.

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	$\star$
2	d_2	0	$+90$	$\star$
3	$\star$	0	0	0
4	0	0	-90	$\star$
5	0	0	$+90$	$\star$
6	d_6	0	0	$\star$

* joint variable

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.36}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.37}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.38}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.39}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.40}$$

T_6^0 is then given as

$$T_6^0 = A_1 \cdots A_6 \tag{3.41}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.42}$$

where

$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \qquad (3.43) \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5). \qquad (3.44)
\end{aligned}$$

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Example 3.6 SCARA Manipulator

As another example of the general procedure, consider the SCARA manipulator of Figure 3.11. This manipulator, which is an abstraction of the AdeptOne robot of Figure 1.11, consists of an RRP arm and a one degree-of-freedom wrist, whose motion is a roll about the vertical axis. The first

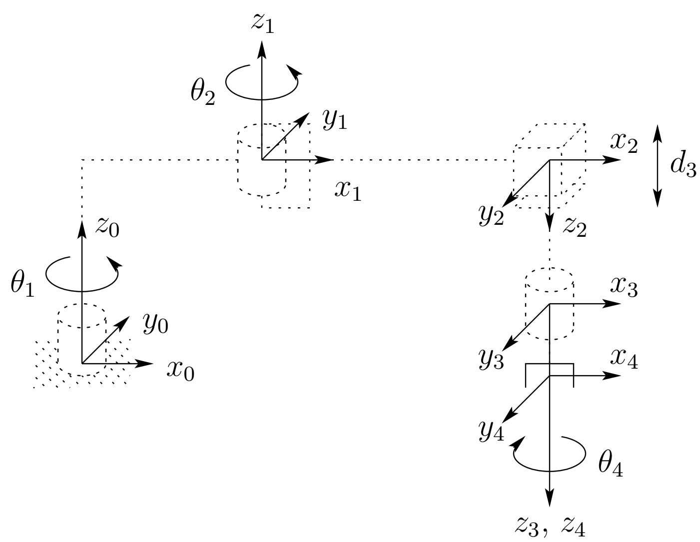

Figure 3.11: DH coordinate frame assignment for the SCARA manipulator.

Table 3.5: Joint parameters for SCARA.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

step is to locate and label the joint axes as shown. Since all joint axes are parallel we have some freedom in the placement of the origins. The origins are placed as shown for convenience. We establish the x_0 *axis in the plane of the page as shown. This is completely arbitrary and only affects the zero configuration of the manipulator, that is, the position of the manipulator when* $\theta_1 = 0$.

The joint parameters are given in Table 3.5, and the A-matrices are as *follows.*

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.45}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.46}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.47}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.48}$$

The forward kinematic equations are therefore given by

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.49}$$

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3.4 Problems

1. Verify the statement after Equation (3.18) that the rotation matrix R has the form (3.16) provided assumptions DH1 and DH2 are satisfied.

2. Consider the three-link planar manipulator shown in Figure 3.12. Derive

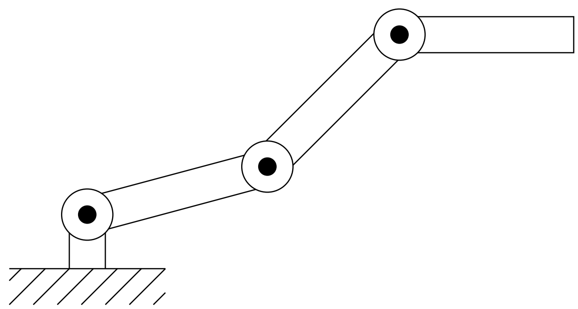

Figure 3.12: Three-link planar arm of Problem 3-2.

the forward kinematic equations using the DH-convention.

3. Consider the two-link cartesian manipulator of Figure 3.13. Derive

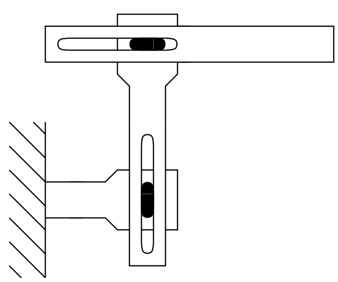

Figure 3.13: Two-link cartesian robot of Problem 3-3.

the forward kinematic equations using the DH-convention.

4. Consider the two-link manipulator of Figure 3.14 which has joint 1 revolute and joint 2 prismatic. Derive the forward kinematic equations using the DH-convention.

5. Consider the three-link planar manipulator of Figure 3.15 Derive the forward kinematic equations using the DH-convention.

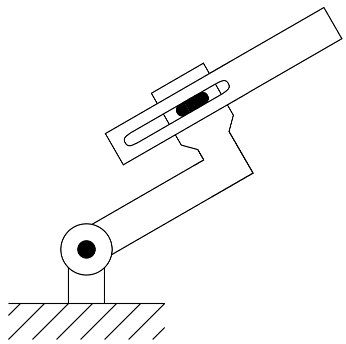

Figure 3.14: Two-link planar arm of Problem 3-4.

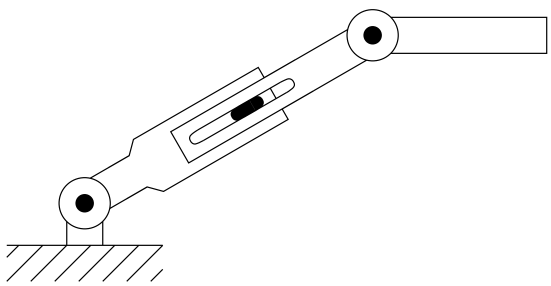

Figure 3.15: Three-link planar arm with prismatic joint of Problem 3-5.

6. Consider the three-link articulated robot of Figure 3.16. Derive the forward kinematic equations using the DH-convention.

7. Consider the three-link cartesian manipulator of Figure 3.17. Derive the forward kinematic equations using the DH-convention.

8. Attach a spherical wrist to the three-link articulated manipulator of Problem 3-6 as shown in Figure 3.18. Derive the forward kinematic equations for this manipulator.

9. Attach a spherical wrist to the three-link cartesian manipulator of Problem 3-7 as shown in Figure 3.19. Derive the forward kinematic equations for this manipulator.

10. Consider the PUMA 260 manipulator shown in Figure 3.20. Derive the complete set of forward kinematic equations, by establishing appropriate D-H coordinate frames, constructing a table of link parameters, forming the A-matrices, etc.

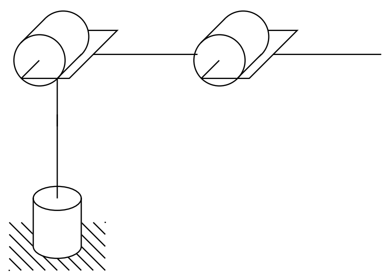

Figure 3.16: Three-link articulated robot.

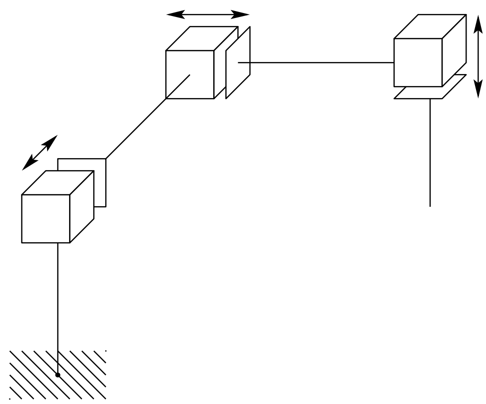

Figure 3.17: Three-link cartesian robot.

11. Repeat Problem 3-9 for the five degree-of-freedom Rhino XR-3 robot shown in Figure 3.21. (Note: you should replace the Rhino wrist with the sperical wrist.)

12. Suppose that a Rhino XR-3 is bolted to a table upon which a coordinate frame $o_s x_s y_s z_s$ is established as shown in Figure 3.22. (The frame $o_s x_s y_x z_s$ is often referred to as the **station frame**.) Given the base frame that you established in Problem 3-11, find the homogeneous transformation T_0^s relating the base frame to the station frame. Find the homogeneous transformation T_5^s relating the end-effector frame to the station frame. What is the position and orientation of the end-effector in the station frame when $\theta_1 = \theta_2 = \cdots = \theta_5 = 0$?

13. Consider the GMF S-400 robot shown in Figure 3.23 Draw the symbolic representation for this manipulator. Establish DH-coordinate frames and write the forward kinematic equations.

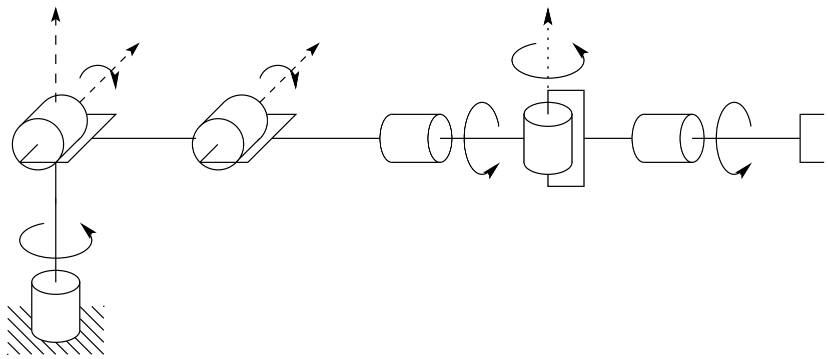

Figure 3.18: Elbow manipulator with spherical wrist.

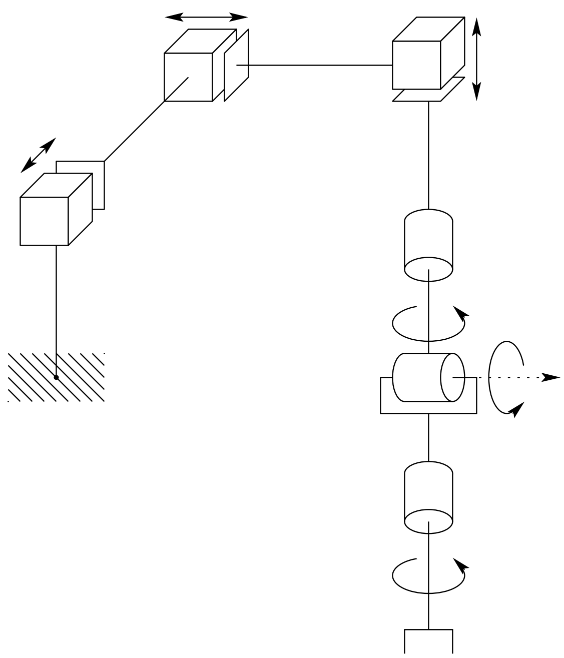

Figure 3.19: Cartesian manipulator with spherical wrist.

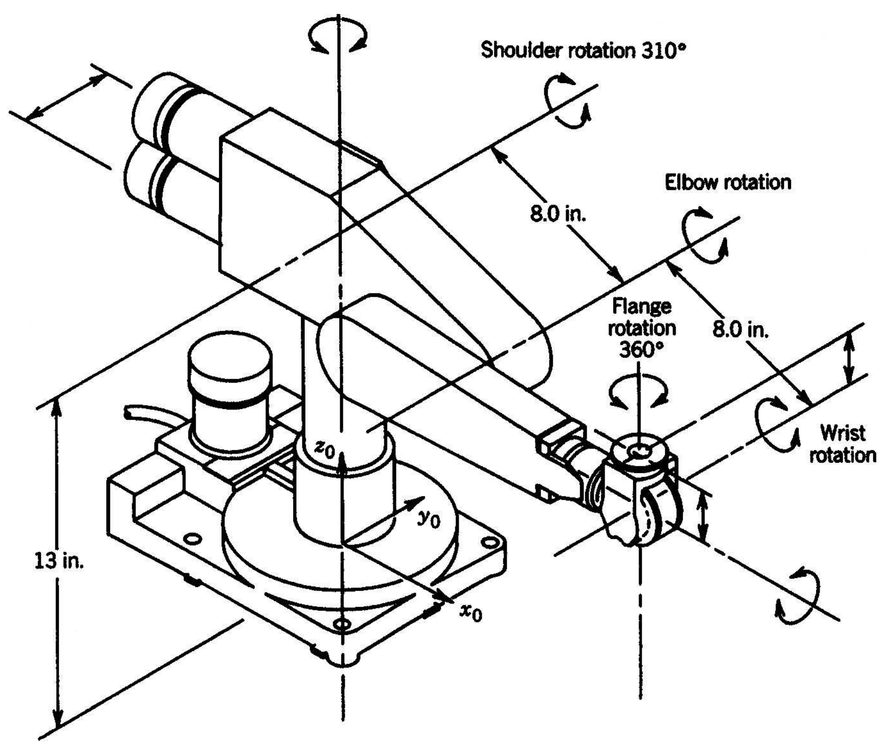

Figure 3.20: PUMA 260 manipulator.

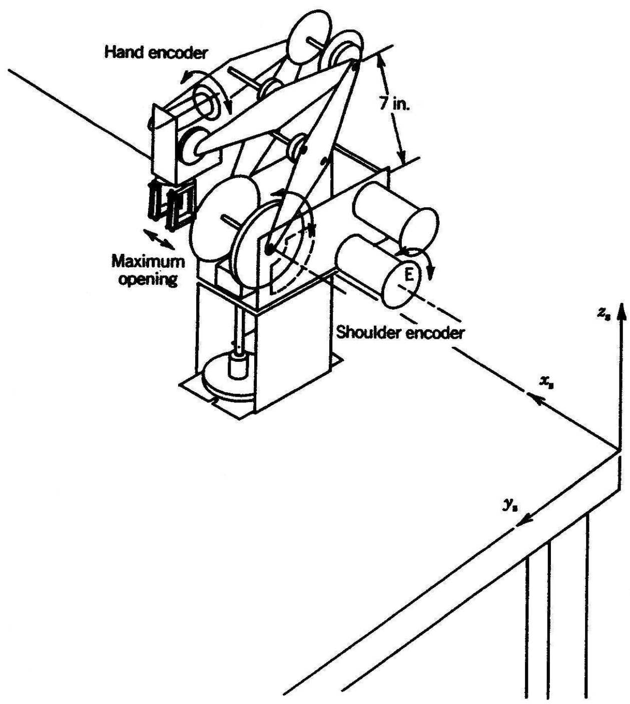

Figure 3.21: Rhino XR-3 robot.

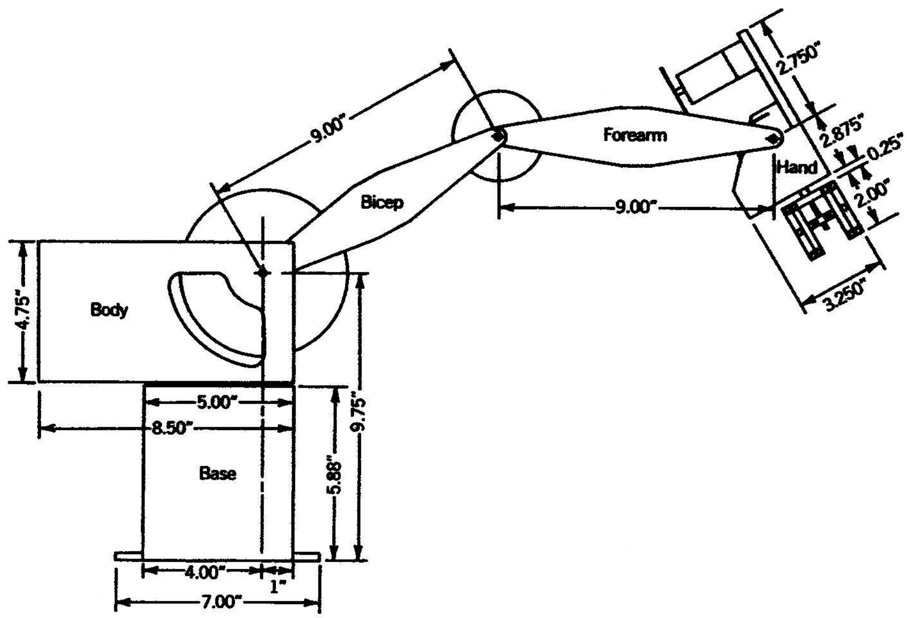

Figure 3.22: Rhino robot attached to a table. From: *A Robot Engineering Textbook*, by Mohsen Shahinpoor. Copyright 1987, Harper & Row Publishers, Inc

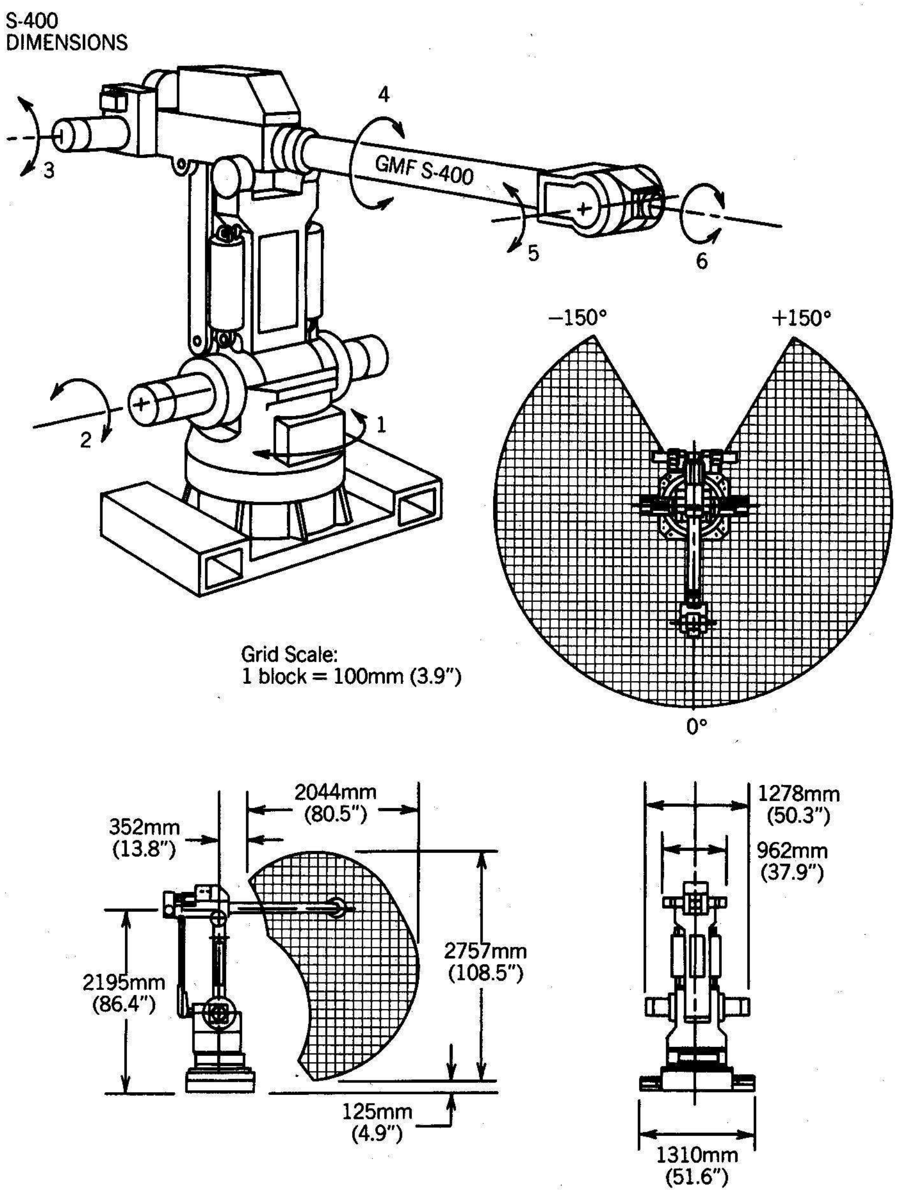

Figure 3.23: GMF S-400 robot. (Courtesy GMF Robotics.)