Principle Component Analysis

Plain Hebb

$$\frac{d\mathbf{w}}{dt} = \mu\,\mathbf{Q}\,\mathbf{w} \tag{1}$$

The correlation matrix is rewritten in eigenvector form with eigenvalue λ_ν and -vector e_ν:

$$\mathbf{Q}\,\mathbf{e}_\nu = \lambda_\nu\,\mathbf{e}_\nu \tag{2}$$

Thus, the weight vector w can be rewritten in Q-space

$$\mathbf{w}(t) = \sum_\nu^N c_\nu(t)\,\mathbf{e}_\nu \tag{3}$$

with coefficients c_ν

$$c_\nu(t) = \mathbf{w}(t)\cdot\mathbf{e}_\nu \tag{4}$$

and we can rewrite Eq. 1 to

$$\sum_\nu^N \frac{dc_\nu}{dt}\,\mathbf{e}_\nu = \mu\,\mathbf{Q}\,\sum_\nu^N c_\nu\,\mathbf{e}_\nu. \tag{5}$$

Thus, with Eq. 2 we get

$$\begin{aligned}
\sum_\nu^N \frac{dc_\nu}{dt}\,\mathbf{e}_\nu &= \mu\,\sum_\nu^N \lambda_\nu\,c_\nu\,\mathbf{e}_\nu \\
\sum_\nu^N \frac{dc_\nu}{dt}\,\mathbf{e}_\nu\cdot\mathbf{e}_\kappa &= \mu\,\sum_\nu^N \lambda_\nu\,c_\nu\,\mathbf{e}_\nu\cdot\mathbf{e}_\kappa \qquad , \mathbf{e}_\nu\cdot\mathbf{e}_\kappa = 0\,\forall\,\nu\neq\kappa \\
\frac{dc_\nu}{dt} &= \mu\,\lambda_\nu\,c_\nu.
\end{aligned}$$

If we solve this differential equation we obtain the development of the coefficients over time

$$c_\nu(t) = c_\nu(0)\,e^{\mu\,\lambda_\nu\,t}. \tag{6}$$

Insert this solution in Eq. 3 and the coefficients (Eq. 4) for $t = 0$:

$$\begin{aligned}
\mathbf{w}(t) &= \sum_\nu^N c_\nu(0)\,e^{\mu\,\lambda_\nu\,t}\,\mathbf{e}_\nu \\
&= \sum_\nu^N \left(\mathbf{w}(\mathbf{0})\cdot\mathbf{e}_\nu\right)\,e^{\mu\,\lambda_\nu\,t}\,\mathbf{e}_\nu.
\end{aligned} \tag{7}$$

As the eigenvalues are rank-ordered ($\lambda_1 > \lambda_2 > ...$) the largest eigenvalue λ_1 with the corresponding eigenvector $\mathbf{e_1}$ will dominate the weight development.

Ocular Dominance in small network

Plain Hebb

The system has the two inputs u_r and u_l and corresponding weights w_r and w_l to one neuron. The correlation matrix has the form:

$$\mathbf{Q} = \begin{pmatrix} q_S & q_D \\ q_D & q_S \end{pmatrix} \tag{8}$$

To calculate the development of the weights one has to consider the eigenvalues and -vectors of the correlation matrix.
Eigenvalue:

$$\begin{aligned} det\,(\mathbf{Q} - \lambda\,\mathbf{1}) &= 0 \\ det\left(\left(\begin{pmatrix} q_S & q_D \\ q_D & q_S \end{pmatrix}\right) - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ det \begin{pmatrix} q_S - \lambda & q_D \\ q_D & q_S - \lambda \end{pmatrix} &= 0 \\ (q_S - \lambda)^2 - q_D^2 &= 0 \\ \Rightarrow \lambda_{1/2} &= q_S \pm q_D \end{aligned}$$

The corresponding eigenvectors are calculated by

$$\begin{aligned} \left[\mathbf{Q} - \lambda_{1/2}\,\mathbf{1}\right] \cdot \mathbf{e_{1/2}} &= \mathbf{0} \\ \left[\begin{pmatrix} q_S & q_D \\ q_D & q_S \end{pmatrix} - \lambda_{1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \begin{pmatrix} a_{1/2} \\ b_{1/2} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Thus, we obtain for λ_1 and λ_2 the following normalized eigenvectors

$$\begin{aligned} \mathbf{e_1} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{e_2} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

The next step is to decouple the differential equation of Hebbian learning (Eq. 1) by changing the coordinate system to the correlation matrix system.

$$\frac{1}{\mu}\frac{dw_r}{dt} = q_S\,w_r + q_D\,w_l \tag{9}$$

$$\frac{1}{\mu}\frac{dw_l}{dt} = q_D\,w_r + q_S\,w_l \tag{10}$$

The first (corresponding to $\mathbf{e_1}$) is to sum Eq. 9 with Eq. 10 and the second (corresponding to $\mathbf{e_2}$) is to subtract Eq. 10 from Eq. 9:

$$\begin{aligned} \frac{1}{\mu}\frac{d(w_r + w_l)}{dt} &= (q_S + q_D) \cdot (w_r + w_l) \\ \frac{1}{\mu}\frac{d(w_r - w_l)}{dt} &= (q_S - q_D) \cdot (w_r - w_l)\,. \end{aligned}$$

With $w_+ = w_r + w_l$ and $w_- = w_r - w_l$ the equations can be reformulated to

$$\frac{1}{\mu}\frac{dw_+}{dt} = \lambda_1 \cdot w_+ \tag{11}$$

$$\frac{1}{\mu}\frac{dw_-}{dt} = \lambda_2 \cdot w_- . \tag{12}$$

Of course the first eigenvalue is larger than the second ($\lambda_1 > \lambda_2$) and, therefore, w_+ or rather $\mathbf{e_1}$ grows faster than $w_-/\mathbf{e_2}$. This means that w_r and w_l grow the same way and no orientation selectivity can occur.

Hebb and multiplicative normalization

The eigenvalues and -vectors of the correlation matrix are not effected by the learning rule. Thus, we can directly start at the decoupling of the differential equations

$$\frac{1}{\mu}\frac{dw_r}{dt} = q_S\, w_r + q_D\, w_l - \alpha v^2\, w_r$$

$$\frac{1}{\mu}\frac{dw_l}{dt} = q_D\, w_r + q_S\, w_l - \alpha v^2\, w_l$$

by transformation (summation and subtraction)

$$\frac{1}{\mu}\frac{dw_+}{dt} = \lambda_1 \cdot w_+ - \alpha v^2 w_+$$

$$\frac{1}{\mu}\frac{dw_-}{dt} = \lambda_2 \cdot w_- - \alpha v^2 w_- .$$

As αv^2 is equal for both terms again the difference between the eigenvalues λ_1 and λ_2 determines the weight growth.

Hebb and subtractive normalization

We again start at the decoupling stage:

$$\frac{1}{\mu}\frac{dw_r}{dt} = q_S\, w_r + q_D\, w_l - \frac{v(\mathbf{n}\cdot\mathbf{u})}{N}$$

$$\frac{1}{\mu}\frac{dw_l}{dt} = q_D\, w_r + q_S\, w_l - \frac{v(\mathbf{n}\cdot\mathbf{u})}{N}.$$

Summation leads to

$$\begin{aligned}
\frac{1}{\mu}\frac{dw_+}{dt} &= \lambda_1 \cdot w_+ - \frac{2}{N}\, v(\mathbf{n}\cdot\mathbf{u}) && , N = 2,\ v = \mathbf{w}\cdot\mathbf{u} \\
&= \lambda_1 \cdot w_+ - (\mathbf{Q}\cdot\mathbf{w})\cdot\mathbf{n} && , \mathbf{Q}\cdot\mathbf{w} = \sum_\nu c_\nu \lambda_\nu e_\nu \\
&= \lambda_1 \cdot w_+ - \left(\frac{1}{\sqrt{2}}\,\lambda_1\, w_+\, \mathbf{e_1} + \frac{1}{\sqrt{2}}\,\lambda_2\, w_-\, \mathbf{e_2}\right)\cdot\mathbf{n} && , \mathbf{e_2}\cdot\mathbf{n} = 0,\ \mathbf{e_1}\cdot\mathbf{n} = \sqrt{2} \\
&= \lambda_1 \cdot w_+ - \lambda_1 \cdot w_+ = 0
\end{aligned}$$

and

$$\frac{1}{\mu}\frac{dw_-}{dt} = \lambda_2 \cdot w_- .$$

Therefore, subtractive normalization guarantees a weight development in $\mathbf{e_2}$ direction. Both weights (w_r and w_l) develop in contrary direction. Who is large and who is small depends on the initial $w_-(0)$.

Network

When do we have stable activity development?

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= -\mathbf{v} + \mathbf{M}\,\mathbf{v} + \mathbf{W}\,\mathbf{u} \\ &= (\mathbf{M} - \mathbf{1}) \cdot \mathbf{v} + \mathbf{W}\,\mathbf{u}.\end{aligned}$$

Solve this equation

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{v}(0)\, e^{(\mathbf{M}-\mathbf{1})\,t} + \mathbf{c}(t) \\ &\approx \mathbf{v}(0)\, e^{(\lambda_\nu - 1)\,t}\, \mathbf{e}_\nu + \mathbf{c}(t)\end{aligned}$$

If all $\lambda_\nu < 1$, the system is stable and we can rewrite the differential equation of the activity for $d\mathbf{v}/dt = 0$ to

$$\begin{aligned}\mathbf{v} &= \mathbf{W}\,\mathbf{u} + \mathbf{M}\,\mathbf{u} \\ \mathbf{v} &= (\mathbf{1} - \mathbf{M})^{-1}\ \mathbf{W}\,\mathbf{u} \\ &= \mathbf{K}\,\mathbf{W}\,\mathbf{u}.\end{aligned}$$

We assume that the recurrent connections $\mathbf{K}$ are constant over time and only the input weights $\mathbf{W}$ are plastic. Thus, we have a similar problem as shown above with a bias $\mathbf{K}$. Assuming again only two inputs u_r and u_l, we can decouple the differential equations as above:

$$\begin{aligned}\frac{1}{\mu}\frac{dw_+}{dt} &= \lambda_1\, K\, w_+ \\ \frac{1}{\mu}\frac{dw_-}{dt} &= \lambda_2\, K\, w_-\end{aligned}$$

If we assume again a subtractive normalization ($dw_+/dt = 0$), the development of w_- depends now on the principle eigenvector of $\mathbf{K}$.

We assume periodic boundaries, thus, **K** is a ciculant matrix with following eigenvalues and -vectors:

$$\begin{pmatrix} k_0 & k_1 & k_2 & \dots & k_{N-1} \\ k_{N-1} & k_0 & k_1 & \dots & k_{N-2} \\ \vdots & & \ddots & & \vdots \\ k_1 & & & \dots & k_0 \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{N-1} \end{pmatrix} = \lambda \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{N-1} \end{pmatrix}$$

This means for each row:

$$\begin{aligned} \lambda\, e_0 &= k_0\, e_0 + k_1\, e_1 + \dots + k_{N-1}\, e_{N-1} \\ &= \sum_{j=0}^{N-1} k_j\, e_j \\ \lambda\, e_1 &= k_{N-1}\, e_0 + k_0\, e_1 + k_1\, e_2 + \dots + k_{N-2}\, e_{N-1} \\ &= k_{N-1}\, e_0 + \sum_{j=1}^{N-1} k_{j-1}\, e_j \\ \lambda\, e_2 &= k_{N-2}\, e_0 + k_{N-1}\, e_1 + k_0\, e_2 + \dots + k_{N-3}\, e_{N-1} \\ &= \sum_{j=0}^{2} k_{N-2-j}\, e_j + \sum_{j=2}^{N-1} k_{j-2}\, e_j \\ \dots \\ \lambda\, e_m &= \sum_{j=0}^{m-1} k_{N-m-j}\, e_j + \sum_{j=m}^{N-1} k_{j-m}\, e_j \\ \text{renumbering leads to} \\ \lambda\, e_m &= \sum_{j=0}^{N-m-1} k_j\, e_{j+m} + \sum_{j=N-m}^{N-1} k_j\, e_{j-N+m} \\ \text{this can be solved by the ansatz}\, e_j &= f^j \\ \lambda f^m &= \sum_{j=0}^{N-m-1} k_j\, f^{j+m} + \sum_{j=N-m}^{N-1} k_j\, f^{j-N+m} \\ \lambda &= \sum_{j=0}^{N-m-1} k_j\, f^j + f^{-N} \sum_{j=N-m}^{N-1} k_j\, f^j \\ \text{if we assume the nth root of unity:}\, f^{-N} &= 1 \\ \lambda &= \sum_{j=0}^{N-1} k_j\, f^j \end{aligned}$$

with eigenvector entries $e_j = 1/\sqrt{N}\, f^j$. Or written different

$$e_a^\omega = e^{i\,\omega\,a} \tag{13}$$

with neuron and entry a and eigenvector/ -value ω.

We can define the eigenvalue problem also in the functional space:

$$
\begin{aligned}
A\, e_\nu &= \lambda_\nu\, e_\nu \\
\Rightarrow \int dt'\, A(t,t')\, e(t') &= \lambda\, e(t).
\end{aligned}
$$

This means for K with eigenvector $\exp(i\omega a)$

$$\tilde{K}(\omega) = \int daK(|a-a'|)\, e^{i\omega\, a}. \tag{14}$$

Interestingly $\tilde{K}$ is the Fourier transformed of K. As we have a discrete number of neurons a, we also have to use the discrete version of the Fourier transformation

$$\tilde{K}(m) = \sum_{a=0}^{N-1} K(|a-a'|)\, e^{i\,2\pi\, a\, m/N}. \tag{15}$$

Thus, with $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$ we get the real part of the ath entry of the mth eigenvector:

$$e_a^m = \cos\left(\frac{2\,\pi\, m\, a}{N} + \Phi\right) \tag{16}$$

As we need to know the maximal eigenvalue to get the principle eigenvector, we have to introduce into Eq. 14 $\Delta = a - a'$

$$\tilde{K}(\omega) = \left(\int d\Delta K(|\Delta|)\, e^{-i\,\omega\,\Delta}\right)\, e^{i\,\omega\, a} = \lambda(\omega)\, e^{i\,\omega\, a}. \tag{17}$$

Thus, $\lambda(\omega)$ is the distribution of eigenvalues. To obtain the maximum ω we have to solve the discrete version of $\tilde{K}$ as the eigenvectors have the length one (by definition, see above)

$$
\begin{aligned}
\tilde{K}(\omega) &= \lambda(\omega)\, e^{i\omega a} && \text{continuous} \\
\Rightarrow \tilde{K}(m) &= \lambda(m)\, e^{i2\pi m a/N} && \text{discrete} \\
\text{re-normalize by } k &= \frac{2\pi m}{Nd} \\
\Rightarrow \tilde{K}(k) &= \lambda(k)\, e^{ikad}.
\end{aligned}
$$

For given K (e.g., two gaussians), the k with the maximal eigenvalue ($\lambda(k)$) can be calculated and set into e^{ikda} to get the direction of the principle eigenvector of K and, therefore, the main direction of ω_-.