Predictive Learning in Rate-Coded Neural Networks: A theoretical approach towards classical conditioning

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Abstract. A novel approach for learning of temporally extended, continuous signals in a neural network is developed. Input signals are band-pass filtered before being summed at an output unit and a new learning rule is devised which utilizes the temporal derivative of the output to modify the weights. The initial development of the weights is calculated within the framework of signal theory and simulation results are shown to demonstrate the performance of this approach. In addition we show that few units suffice to process multiple inputs with long temporal delays.

1 Introduction

Hebb like learning is one of the basic ideas of computational neuroscience. This learning causes changes of the synaptic weights between (artificial) neurons according to post- and presynaptic correlations. The strength is increased with simultaneous activity and decreased with asynchronous activity. Important in the context of this paper is that this learning rule is symmetric in time. The learning rule doesn't discriminate which if the postsynaptic spike is earlier or later than the presynaptic spike. In contrast to this the temporal Hebb rule is asymmetric in time: A weight will be strengthened only if the input precedes the output by a short interval. If the order of input and output is reversed the weight will be decreased [1–3]. While the classical Hebb rule has it's applicatons in the field of spatial pattern formation and detection [3] the temporal Hebb rule is dealing with temporal patterns called time sequences (for a review see [4]).

Recent theoretical approaches towards temporal learning rest on on spiking neurons [2]. These models are often highly realistic but usually elude from a rigorous mathematical analysis. Also one finds that long time scales are hard to treat with these models, so that the resultig applications are limited to the detection of small time differences [5].

On the other hand there exists a long tradition of models with linear or so called rate coded neurons which allow an easier treatment [6]. Thus in our approach we have chosen a rate code description as the level of abstraction and present a theoretical framework for predictive (temporal) learning which is able to handle time-continuous input signals (rate-functions) of arbitrary shape. To this end our "neurons" will act as damped oscillator circuits and a new learning rule is developed which utilizes the temporal change (the derivative) of the output to modify the weights. This system has the feature, that very few components are required to cover large time intervals and it allows to use multiple inputs, which are the two critical prerequisites for "classical conditioning".

2 Definition and function of the neuronal circuit

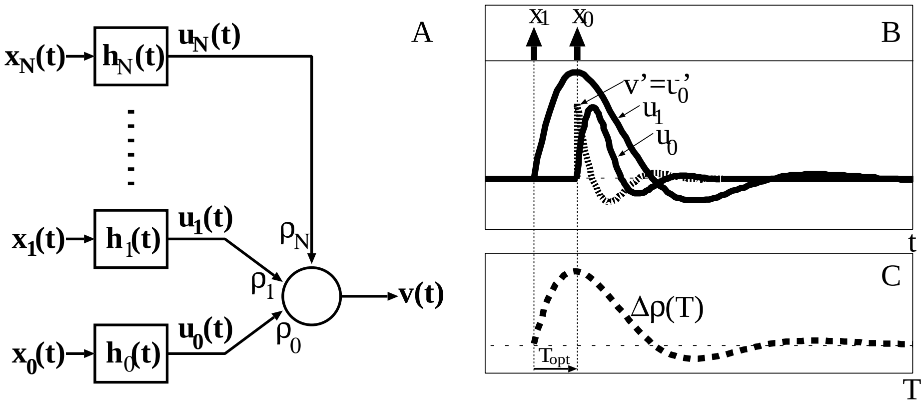

Fig. 1. A) The basic circuit in the time domain. B) Input functions and the initial weight change for $t = 0$ according to Eq. 19. A shows the inputs x, the impulse responses u for a choice of two different resonators h and the derivative of the output v'. B shows the initial weight change $\rho_1(T)_{t=0}$. The delay between x_0 and x_1 is adjusted to $T = \frac{\pi}{2b_1}$ leading to $\rho_1(T) = max$. Thus, this setup represents the optimal solution, $T = T_{opt}$, for the given resonator h_1.

We consider a system of N units h receiving a continuous input signal x and producing an continous output u. The input units connect with weights ρ to one output unit v (Fig. 1A). All input units are in principle equivalent but we will use h_0 to denote the one unit which transmits the unconditioned stimulus. The output v is then given as:

$$v = \rho_0 u_0 + \sum_{i=1}^{N} \rho_i u_i \qquad \text{with} \tag{1}$$

$$u_i(t) = x_i(t) * h_i(t) \tag{2}$$

The transfer function h shall be that of a *resonator* which transforms a δ-pulse input into a damped oscillation. The response of such a resonator to a δ-pulse is given by (see Fig. 1B).

$$h(t) = \frac{1}{b} e^{-at} \sin(bt) \tag{3}$$

Learning (viz. weight change) takes place according to a Hebb-like rule (Eq. 5):

$$\rho_i \rightarrow \rho_i + \Delta\rho_i \qquad i = 1, 2, 3... \tag{4}$$

$$\Delta\rho_i \quad = \mu u_i v' \qquad \mu \ll 1 \tag{5}$$

where the weight change depends on the correlation between u_i and the derivative of v (see Fig. 1B). As usual we require that all weight changes occur on a much longer time scale (i.e., very slowly) as compared to the decay of the oscillatory responses u. This allows us to treat the system in a steady state condition, thus, $\Delta\rho \rightarrow 0$ for $\Delta t \rightarrow 0$.

> We state our goal as: After learning, the output unit shall produce a well discernible signal v (e.g., of high amplitude and steeply rising) in response to the earliest occurring conditioned stimulus $x_j, j \geq 1$.

3 Analytical solution of the initial weight change

Similar to other approaches [7] we compute the initial development of the weights as soon as learning starts, because this is indicative of the continuation of the learning process. In order to calculate this weight change we will now introduce several restrictions:

1. The weight of the unconditioned stimulus is set to $\rho_0 = 1$ and kept constant throughout learning.
2. We will consider only one unit that can learn, thus, $N = 1$.
3. Accordingly we have to deal with only two input functions x_0, x_1 and we define them as (delayed) δ-pulses:

$$x_0(t) = \delta(t + T), \qquad T \geq 0 \tag{6}$$

$$x_1(t) = \delta(t) \tag{7}$$

These restrictions will allow us to develop the theory but can be waived in the end without affecting our basic findings.

Because we assume steady-state, we can rewrite the product in the learning rule (Eq. 5) as a correlation integral between input and output:

$$\Delta\rho_1(T) = \mu \int_0^\infty u_1(T + \tau) v'(\tau) d\tau \tag{8}$$

The value of T represents the delay between conditional and unconditional stimulus. In order to assure optimal learning progress we must, therefore, determine that particular value T_{opt} for which a maximal weight change is observed. Because we are interersted in the initial weight change, we can assume:

$$\rho_1(t) = 0 \qquad \text{for} \quad t = 0 \tag{9}$$

and Eq. 8 turns into:

$$\Delta\rho_1(T)_{t=0} = \mu \int_0^{\infty} u_1(T+\tau)u_0'(\tau)d\tau \tag{10}$$

In simple cases (e.g., for $h_0 = h_1$) this integral can be solved directly. A general solution, which can also be extended to cover more than two inputs, requires to apply the LAPLACE transform using the notational convention: $x(t) \leftrightarrow X(s)$, for a transformation pair of functions in the time and the LAPLACE domain.

For the inputs we get:

$$x_0(t) = \delta(t+T) \leftrightarrow X_0(s) = e^{-Ts} \tag{11}$$

$$x_1(t) = \delta(t) \leftrightarrow X_1(s) = 1 \tag{12}$$

Then we specify the resonators in the usual way by:

$$h(t) \leftrightarrow H(s) = \frac{1}{(s-p)(s-p^*)} \tag{13}$$

where p^* represents the complex conjugate of the pole p. It is important to note that such a resonator is only stable if its pole-pair is located on the left complex half-plane, otherwise an amplified oscillation is obtained.

Real and imaginary parts of the poles are given by $a := \mathrm{Re}(p) = \pi f/Q$ and $b := \mathrm{Im}(p) = \sqrt{(2\pi f)^2 - a^2}$, where f is the frequency of the oscillation. The damping characteristic of the resonator is reflected by $Q \geq 0.5$. Small values of Q lead to a strong damping.

In the general case we will have two different resonators $H_0(s)$ and $H_1(s)$ with different conjugate pole pairs (p_0, p_0^*) and (p_1, p_1^*) and we get for the responses:

$$U_0(s) = e^{-sT}H_0(s) \tag{14}$$

$$U_1(s) = H_1(s) \tag{15}$$

In the LAPLACE domain, the derivative of U_0 is simply a multiplication with s:

$$v'(t) = u_0'(t) \leftrightarrow sV(s) = sU_0(s) = se^{-sT}H_0(s) \tag{16}$$

In order to compute the weight change (Eq. 10) we use *Plancherel's* theorem and get:

$$\Delta\rho_1 = \mu\frac{1}{2\pi}\int_{-\infty}^{+\infty} H_1(-i\omega)\left[i\omega e^{-Ti\omega}H_0(i\omega)\right]d\omega \tag{17}$$

$$= \mu\frac{1}{2\pi}\int_{-\infty}^{+\infty} H_1(i\omega)\left[-i\omega e^{Ti\omega}H_0(-i\omega)\right]d\omega \tag{18}$$

Note that symmetry of the *Plancherel's* theorem is broken due to the exponential term. Equation 17 represents a FOURIER transform and Eq. 18 its inverse. Both integrals can be evaluated with the method of residuals. Eq. 18, however, offers the advantage that we can neglect the right complex half plane, because it leads to contributions for negative time (i.e. $T < 0$) only [8, 9]. Thus, of the four residuals (poles) for H_1 and H_0 only those of H_1 need to be considered because those of H_0 have flipped their sign to the right side in Eq. 18. We get as the final result:

$$\Delta\rho_1(T)_{t=0} = \frac{1}{4a_1 b_1}\sin(b_1 T)e^{-a_1 T} \tag{19}$$

The initial weight change (viz. for $t = 0$) depends only on h_1, which is a reasonable result given that at $t = 0$ no influence of the unconditioned stimulus can be felt (Fig. 1B). Before showing simulation results we note that the above obtained analytical results can be extended to cover the most general system structure as represented in Fig. 1A. Equation 1 turns into:

$$V(s) = \sum_{j=0}^{N} \rho_j e^{-sT_j} U_j(s) \qquad \text{for} \quad T_j \geq 0 \tag{20}$$

keeping it in the LAPLACE domain, because then we can directly obtain:

$$\Delta\rho_i(T) = \mu \frac{1}{2\pi} \int_{-\infty}^{+\infty} -i\omega V(-i\omega) U_i(i\omega) d\omega, \tag{21}$$

which is the general form of Eq. 18. It should be noted that for all $\Delta\rho_i$ this integral can still be evaluated analytically in the same way as in the special case with two resonators discussed above.

4 Simulations

In order to validate the approach several simulations of increasing complexity were performed. The analytical solution treats only the initial learning step, i.e. $t = 0$. Thus, first we used the same setup as before and determined the learning behavior for $t >> 0$. Fig. 2 A shows that at the initial learning step the output function v still coincides with the unconditioned stimulus response u_0. After 10 repetitions of the pulse-sequence (Fig. 2 B), the output function has shifted forward to u_1. This forward shift can be interpreted in the sense of classical conditioning: after learning the output signal v predicts the occurrence of u_0 having been conditioned by u_1.

A sufficiently strong correlation occurs in this setup only for small temporal difference between the input pulses. To improve on this one can use several resonators $h_1, ..., h_n$ with different frequencies which will all receive input from the conditioning stimulus x_1, defining: $T_0 = T; x_j = x_1, T_j = 0$ for $j \geq 1$ (Fig. 1A and Eq. 20). In Fig. 2 C-E we use five resonators to represent x_1 and show the results after 10 learning steps for increasingly larger intervals T which separate the input pulses. First we note that in all cases the output is a superposition of the different resonator outputs and that a well discernible early signal component exists in response to the conditioning stimulus x_1. This is due to the fact that in this setup the learning process can be seen as a sequential improvement of the

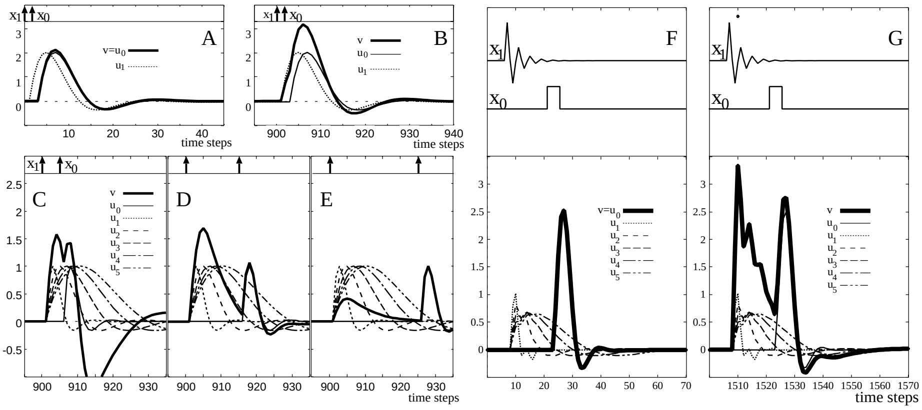

Fig. 2. Simulation results. Arbitrary time steps were used in the simulations. Pulse sequences were repeated every 100 time-steps, the first starting at zero. Resonators were implemented digitally as IIR-filters, which leads to a small onset-delay of 2 time-steps. They had always a value of $Q = 1$. A,B) are obtained with two identical resonators h_0, h_1 with: $f_{0,1} = 0.1$. The other parameters were $\mu = 0.01$ and $T = 2$. A) Result for $t = 0$, B) for $t = 900$. C-E) are calculated with six resonators, five (curves of $u_1, ..., u_5$) receiving input from x_1 and one (curve of u_0) from x_0. Resonator parameters are: $f_0 = 0.1$, $f_j = \frac{f_0}{j}, j \geq 1$, μ was set to 0.11. Results are shown for $t = 900$. C) $T = 5$. D) $T = 15$, E) $T = 25$. F) and G) have the same conditions like D) but for x_1 a sine burst is used.

rising edge at the onset of the conditioned stimulus. Schematically: Correlation first takes place between $v = u_0$ and u_5. By this v (hence also v') shifts forward to some degree and in the next step the correlation can spread to the higher frequency resonator response(s), and so on. Learning, however, needs more and more steps when longer durations of T are used. This leads to a still rather feeble early response component in E, which only improves after a few more iterations.

The use of multiple resonators at a first glance resembles the afore criticized, commonly used strategy of designing multiple delay lines to treat long temporal difference in sequence learning [4]. However, on theoretical grounds, two resonators will in principle always suffice to cover arbitrarily long time intervals T, when setting the wave-length of h_1 to $4T$ and thus the imaginary part of p_1 to $b_1 = \frac{\pi}{2T}$ (see Eq. 19). The shape of the curve v will then at early learning steps be similar to that in Fig 2 E. Using two resonators may, however, lead to impracticable solutions (e.g., slow convergence of learning).

In Fig 2 E/F we replaced the square pulse of x_1 by a sine burst in order to get a more realistic situation and to show that the system is able to deal with arbitrary input signals. Here the signal x_0 (the unconditioned stimulus) could be a light stimulus and the signal x_1 (the conditioned stimulus) a sound event. After learning the system is again able to detect the sine burst as a predictor of square pulse in signal x_0.

5 Discussion

From the above results it can be deduced that the system generalizes without problems to more generic input combinations (e.g. more than two inputs). In addition, we note that our approach, which has been developed within the framework of the signal theory, allows to waive the restriction to δ-function inputs, because of the sampling properties of time-continuous signals. An important feature of other algorithms for sequence learning is that after convergence the output function will approximate an "expectation potential", which is essentially a rectangular function, starting at x_1 and ending at x_0 (see [4], chapter 10). This is the reason, why our theory was developed using resonators (band-passes), because they allow to compose arbitrary shapes of v, such as expectation potentials. We realize, however, that a pure low-pass characteristic would also suffice to induce temporal learning. General signal composition, however, would then be prevented. The inherently present low-pass component in every resonator relates our approach to the theory of adaptive filters [10]. This helps to understand the predictive properties of our system, because from this context it is known that the derivative of a low-pass filtered signal acts as a predictor of the signal [11]. However, adaptive filters rely on the (gradually shifting) self-similarity of a single signal - hence on its auto-correlation properties. Whereas our approach gains its predictive power from the cross-correlation properties between two (or more) signals such as in biological classical conditioning. Therefore, we expect that this approach will also prove useful in a more technologically oriented context.

Acknowledgments

The authors thank Leslie Smith for his comments on the manuscript.

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